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Gaussian Limit for Critical Oriented Percolation in High Dimensions

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In this paper, we consider the spread-out oriented bond percolation models in $Z^d \times Z$ with $d > 4$ and the nearest-neighbor oriented bond percolation model in sufficiently high dimensions. Let η_n, $n = 1, 2,...,$ be the random measures defined on R^d by

$$\eta_n(A) = \sum_{x \in Z^d} \mathbf{1}_A(x/\sqrt{n}) \mathbf{1}_{\{(0,0) \to (x,n)\}}$$

The mean of η_n, denoted by $\bar{\eta}_\eta$, is the measure defined by

$$\bar{\eta}_n(A) = E_p[\eta_n(A)]$$

We use the lace expansion method to show that the sequence of probability measures $[\bar{\eta}_n(R^d)]^{-1}\bar{\eta}_n$ converges weakly to a Gaussian limit as $n \to \infty$ for every p in the subcritical regime as well as the critical regime of these percolation models. Also we show that for these models the parallel correlation length $\xi(p) \sim |p_c - p|^{-1}$ as $p \uparrow p_c$.

KEY WORDS: Oreinted percolation; connectivity function; lace expansion; infrared bound; Gaussian limit; critical exponent; parallel correlation length; mean-field behavior.

1. MOTIVATION AND MAIN RESULTS

Consider the independent Bernoulli oriented bond percolation models defined on the $(d+1)$-dimensional lattice $Z^d \times Z$ as follows. Draw for each ordered pair of sites $\{(x, n), (y, n+1)\}$ in $Z^d \times Z$ an oriented bond from (x, n) to $(y, n+1)$. Let each oriented bond be independently open with

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probability $p_{x,y}$ and closed with probability $1-p_{x,y}$. We assume the translation invariance and the symmetry by requiring that $p_{x,y}=p_{y,x}=p_{0,y-x}$. Main results in this paper will be devoted to the following situations:

(I) Nearest-neighbor model:

$$p_{0,x}=\begin{cases} p & \text{if } |x|=1 \\ 0 & \text{otherwise} \end{cases}$$

where $|x|$ is the Euclidean length of x.

(II) Spread-out models: $p_{0,x}=pg(x/L)\,L^{-d}$, where the function g is nonnegative and is defined on R_d such that

1. $\int_{R^d} g(x)\,dx=1$.
2. g has compact support.
3. g is invariant under rotations by $\pi/2$ and reflections in the coordinate hyperplanes.
4. $\partial^I g(x)$ is piecewise continuous and $\int |\partial^I g(x)|\,dx<\infty$.
5. $g(x)$, $\partial^I g(x)$ are continuous at 0 and $g(0)>0$.

Here ∂^I means $\prod_{\mu\in I}\partial_\mu$ for $I\subset\{1,2,...,d\}$ and is interpreted as a ditribution. Notice that the spead-out models include the ones in which $g(x)=c_d\mathbf{1}_{\{|x|\leqslant 1/2\}}$ [here c_d is a normalization constant such that $\int_{R^d} g(x)\,dx=1$]; i.e., in terms of their bond densities.

$$p_{0,x}=\begin{cases} pc_dL^{-d} & \text{if } |x|\leqslant L/2 \\ 0 & \text{otherwise} \end{cases}$$

It is commonly believed that the spread-out models belong to the same universality class as the nearest-neighbor model.

The probability and the expectation of the model with bond density $\{p_{0,x}\}$ will be denoted by P_p, E_p, respectively. Declare that (x,n) can be reached from (y,m) if there is an open path connecting (x,n) from (y,m); i.e., there is a sequence of sites $(u_0,n_0)=(y,m)$, (u_1,n_1),..., $(u_j,n_j)=(x,n)$ such that for each $i=1,...,j$ the bond $\{(u_{i-1},n_{i-1}),\ (u_i,n_i)\}$ is open. Denote this event by $\{(y,m)\to(x,n)\}$. For convenience, we denote the connectivity function $P_p((0,0)\to(x,n))$ by $\varphi(x,n)$ and assume that $\varphi(0,0)=1$; i.e., a site is connected to itself. We also use $\Psi(x,n)$ to denote $\varphi(x,n)-\mathbf{1}_{(0,0)}(x,n)$, where $\mathbf{1}_{(y,m)}(x,n)$ means 1 if $(y,m)=(x,n)$ and 0 otherwise. We then introduce the Fourier–Laplace transform (here and from now on we use $\sum_{(x,n)}$ to denote $\sum_{(x,n)\in Z^d\times Z}$ whenever there is no ambiguity)

$$\hat{\varphi}(k,u+it)=\sum_{(x,n)} e^{n(u+it)}e^{ik.x}\,\varphi(x,n) \tag{1}$$

for every $(k, t) \in [-\pi, \pi]^d \times [-\pi, \pi]$ and $u \in (-\infty, m_p)$, where e^{m_p} is the radius of convergence of the power series, expanded with respect to e^z, $z = u + it$, of $\hat{\varphi}(0, z)$. Clearly $\hat{\varphi}(k, z) = \hat{\Psi}(k, z) + 1$.

The quantity m_p has been known as the mass gap or the inverse of the parallel correlation length $\xi(p)$, which is defined by

$$\xi(p) = -\lim_{n \to \infty} \frac{n}{\log Z_n(0)} \tag{2}$$

where $Z_n(k) = \sum_{x \in Z^d} e^{ik.x} \varphi(x, n)$. From (2) it follows that $e^{1/\xi(p)}$ is the radius of convergence of the series defined by $\hat{\varphi}(0, u)$.

Note that the limit on the right-hand side of the above equation exists due to the following submultiplicative property:

$$Z_n(0) \leqslant Z_m(0)\, Z_{n-m}(0) \qquad \text{for every} \quad m \in \{0, 1, 2,..., n\} \tag{3}$$

[To show (3) we observe that if $(0, 0) \to (x, n)$, then for any $m \in \{0, 1,..., n\}$ there exists a $(y, m) \in Z^d \times Z$ such that $(0, 0) \to (y, m)$ and $(y, m) \to (x, n)$; hence,

$$P_p((0, 0) \to (x, n)) \leqslant \sum_{y \in Z^d} P_p((0, 0) \to (y, m))\, P_p((y, m) \to (x, n))$$

We then sum over $x \in Z^d$ both sides of the above inequality to obtain (3).]

Let $C_0 = \{(x, n) \in Z^d \times Z: (0, 0) \to (x, n)\}$ and set $|C_0|$ = the number of sites in C_0. Using the Fubini theorem, we can see that

$$\begin{aligned} E_p(|C_0|) &= E_p\left(\sum_{(x,n)} \mathbf{1}_{\{(x,n) \in C_0\}} \right) \\ &= \sum_{(x,n)} E_p(\mathbf{1}_{\{(x,n) \in C_0\}}) \\ &= \hat{\varphi}(0, 0) \end{aligned} \tag{4}$$

Since $E_p(|C_0|)$ is nondecreasing with respect to p, we can define the critical point

$$p_c = \sup\{p: E_p(|C_0|) < \infty\}$$

Notice that we only consider $p \in (0, p_{\max}]$, where

$$p_{\max} \equiv L^d[\sup\{g(x): x \in Z^d\}]^{-1}$$

for the trivial reason that $\sup\{p_{0,x}: x \in Z^d\} \leqslant 1$. It is not hard to show that $p_c < p_{\max}$. Lower bounds for p_c can be obtained, as we can see below.

For the spread-out models, by overcounting the paths connecting (x, n) from $(0, 0)$ we can show that

$$\varphi(x, n) \leqslant (p/p_L)^n \, G(x, n) \tag{5}$$

where

$$p_L \equiv \left[\sum_{x \in Z^d} g(x/L)\, L^{-d} \right]^{-1}$$

and $G(x, n)$ is the n-step transition function $\text{Prob}(S_n = x)$ of the random walk $S_n = \sum_{i=1}^{n} X_i$ in which X_i are i.i.d. with

$$E(e^{ik.X_1}) = \hat{D}_L(k) \equiv p_L \sum_{x \in Z^d} g(x/L)\, L^{-d} e^{ik.x}$$

Suming both sides of (5) over $(x, n) \in Z^d \times Z$, we have for $p < p_L$

$$\sum_{(x,n)} \varphi(x, n) \leqslant \sum_{(x,n)} (p/p_L)^n \, G(x, n) < \infty \tag{6}$$

Then (6) shows that $p_L \leqslant p_c$.

Similarly, for the nearest-neighbor model, if $p \leqslant 1/(2d)$ then we also have

$$\varphi(x, n) \leqslant (2dp)^n \, G^{n.n.}(x, n)$$

where $G^{n.n.}(x, n)$ is the n-step transition function of the simple random walk; hence, $1/(2d) \leqslant p_c$.

Remark 1. It is not hard to use the results of ref. 17 to show that $p_c = 1/(2d) + O(1/d^2)$, which is, however, weaker than the result shown by Cox and Durrett[7] for an analogous nearest-neighbor oriented bond percolation model on Z^d [in this model each site $x \in Z^d$ is branching out to d positively oriented bonds $(x, x + e_\mu)$, where $\{e_\mu; \mu = 1,..., d\}$ are d canonical unit vectors in Z^d, instead of $2d$ bonds, and each oriented bond $(x, x + e_\mu)$ is independently open with probability p and closed with probability $1 - p$] that the critical probability $p_c(d)$ behaves asymptotically as

$$d^{-1} + \tfrac{1}{2} d^{-3} + o(d^{-3}) \leqslant p_c(d) \leqslant d^{-1} + d^{-3} + O(d^{-4})$$

It is easy to see that if $m_p > 0$, then $E_p(|C_0|) < \infty$. The converse of this can be shown by using the Simon–Lieb correlation inequalities for percolation as discussed in ref. 8. Thus

$$m_p > 0 \qquad \text{iff} \quad E_p(|C_0|) < \infty \tag{7}$$

In addition, one can apply Aizenman and Newman's technique[2] to show

$$E_p(|C_0|) = \infty \qquad \text{and} \qquad m_p = 0 \qquad \text{if} \quad p = p_c \tag{8}$$

Furthermore, it is well known from ref. 1 or ref. 16 that

$$p_c = \inf\{p: P_p(|C_0| = \infty) > 0\} \tag{9}$$

and from ref. 4 that

$$P_p(|C_0| = \infty) = 0 \qquad \text{if} \quad p = p_c$$

In our earlier paper[17] we applied the lace expansion method to obtain the Aizenman–Newman triangle condition (see refs. 2 and 3 for discussions on the triangle condition) and the following mean-field behavior.

Theorem 1. For the nearest-neighbor independent Bernoulli oriented bond percolation model defined on $Z^d \times Z$ with sufficiently high dimensions $d \geqslant d_0$ and for the spread-out bond percolation models defined on $Z^d \times Z$, $d > 4$, with sufficiently large $L \geqslant L_0$, we have

$$E_p(|C_0|) \sim (p_c - p)^{-1} \qquad \text{as} \quad p \uparrow p_c$$

$$P_p(|C_0| = \infty) \sim (p - p_c)^1 \qquad \text{as} \quad p \downarrow p_c$$

$$\sum_{1 \leqslant n \leqslant \infty} P_{p_c}(|C_0| = n)(1 - e^{-nh}) \sim h^{1/2} \qquad \text{as} \quad h \downarrow 0$$

$$E_p(|C_0|^{t+1})/E_p(|C_0|^t) \sim (p_c - p)^{-2} \qquad \text{as} \quad p \uparrow p_c$$

where $E_p(|C_0|) \sim (p_c - p)^{-\gamma}$ as $p \uparrow p_c$ means that there are positive constants K_1, K_2 such that $K_1(p_c - p)^{-\gamma} \leqslant E_p(|C_0|) \leqslant K_2(p_c - p)^{-\gamma}$, and similarly for the others.

It was predicted by many physicists[18,9] that, in addition to the critical behavior described in Theorem 1, the critical exponent defined by $\xi(p)$ also takes the mean-field value, i.e., as $p \uparrow p_c$

$$\xi(p) \sim |p_c - p|^{-1} \tag{10}$$

for the nearest-neighbor (and for the one that is of the same universality class; e.g., the spread-out) oriented percolation model in dimensions $d > 4$. The following theorem shows that this is indeed the case for the models described therein.

Theorem 2. For the spread-out independent Bernoulli bond oriented percolation models defined on $Z^d \times Z$ with $d > 4$ and sufficiently large $L \geqslant L_1 \geqslant L_0$ we have the following infrared bound:

$$|\hat{\varphi}(k, m_p - s + it)| \leqslant \frac{1}{c_1 L^2 |k|^2 + c_2 |t| + c_3 |s|} \tag{11}$$

where c_1, c_2, c_3 are strictly positive constants, uniformly with respect to $p \in (0, p_c]$, $(k, t) \in [-\pi, \pi]^d \times [-\pi, \pi]$, and $s \in (0, 1]$.

Also for the nearest-neighbor oriented bond percolation model in sufficiently high dimensions $d \geqslant d_1 \geqslant d_0 > 4$, the same bound holds with c_1, c_2, c_3, L replaced by c_1', c_2', c_3', 1, respectively.

To see why Theorem 2 implies (10), we insert $s = m_p$, $k = 0$, $t = 0$ in (11) to obtain, for any p in a sufficiently small neighborhood below p_c,

$$E_p(|C_0|) = \hat{\varphi}(0, 0) \leqslant \frac{1}{c_3 m_p} \tag{12}$$

On the other hand, from the submultiplicative property of $Z_n(0)$ we have

$$e^{-nm_p} \leqslant Z_n(0) \tag{13}$$

Summing over n the above inequality, we obtain for $0 < p < p_c$

$$\frac{1}{1 - e^{-m_p}} = \sum_{n=0}^{\infty} e^{-nm_p} \leqslant \sum_{n=0}^{\infty} Z_n(0) = E_p(|C_0|) \tag{14}$$

Observing from Theorem 1 that $E_p(|C_0|) \sim |p_c - p|^{-1}$ as $p \uparrow p_c$, we obtain (10) from (14) and (12) since $(1 - e^{-m_p})^{-1} \geqslant \text{const} \cdot m_p^{-1}$.

Remark 2. The bound (11) is an improved version of inequality (5) of our earlier paper.[17]

Remark 3. The parallel correlation length of oriented percolation is not analogous to the usual correlation length of unoriented percolation, because the isotropic property holds only for the unoriented system. In fact, it was shown by Hara[10] that the correlation length of the connectivity function of unoriented percolation behaves as $|p - p_c|^{-1/2}$ when $p \uparrow p_c$.

Furthermore, we define

$$\eta_n(f) = \sum_{x \in Z^d} f(x/\sqrt{n}) \, \mathbf{1}_{\{(0,0) \to (x,n)\}}$$

for every bounded Borel function $f: R^d \to R$. We think of η_n as the random measures defined by $\eta_n(A) \equiv \eta_n(1_A)$ for every Borel $A \subset R^d$. The mean of η_n, denoted by $\bar{\eta}_n$, is the masure defined by

$$\bar{\eta}_n(A) = E_p[\eta_n(A)]$$

for every Borel $A \subset R^d$. We are interested in the asymptotic behavior of the renormalized probability measures $[\bar{\eta}_n(R^d)]^{-1} \bar{\eta}_n$ as $n \to \infty$.

In ref. 21, using the lace expansion method, we showed that for every $p \in (0, 1/4d]$ of the nearest-neighbor oriented percolation model in high dimensions the probability measures $[\bar{\eta}_n(R^d)]^{-1} \bar{\eta}_n$ converge weakly to a Gaussian limit as $n \to \infty$. We now extend the analysis discussed in our earlier papers[17,21] to show the following theorem.

Theorem 3. Consider the percolation models mentioned in Theorem 2. Then the following statement holds:

For any given $p \in (0, p_c]$ there exist strictly positive constants A_0, A_1 depending upon p such that as $n \to \infty$:

$$\text{(a)}\quad Z_n(0) = \frac{e^{-nm_p}}{A_0 e^{m_p}} [1 + O(n^{-\varepsilon})] \quad \text{for any} \quad \varepsilon \in \left(0, \frac{1}{2}\right)$$

$$\text{(b)}\quad [Z_n(0)]^{-1} \sum_{x \in Z^d} |x|^2 \varphi(x, n) = \frac{2dA_1}{A_0 e^{m_p}} n[1 + O(n^{-\varepsilon})] \quad \text{for any} \quad \varepsilon \in \left(0, \frac{1}{2}\right)$$

$$\text{(c)}\quad [Z_n(0)]^{-1} Z_n\left(\frac{k}{\sqrt{n}}\right) \to \exp\left(-\frac{A_1}{A_0 e^{m_p}} |k|^2\right)$$

Note that since

$$\bar{\eta}_n(e^{ik.x}) \equiv \sum_x \exp(ix.k/\sqrt{n})\, \varphi(x, n) = Z_n(k/\sqrt{n})$$

$$\bar{\eta}_n(R^d) \equiv \sum_x \varphi(x, n) = Z_n(0)$$

so the measures $[\bar{\eta}_n(R^d)]^{-1} \bar{\eta}_n$ converge weakly to a Gaussian limit.

It turns out that in proving Theorems 2 and 3 we need to analyze the connected part $\Psi_c(x, n)$, whose Fourier–Laplace transform $\hat{\Psi}_c(k, z)$ is defined via the renewal equation

$$\hat{\varphi}(k, z) = \frac{1 + \hat{\Psi}_c(k, z)}{1 - pp_L^{-1} e^z \hat{D}_L(k)[1 + \hat{\Psi}_c(k, z)]} \tag{15}$$

uniformly for $0 < p \leqslant p_c$, $\mathrm{Re}(z) < m_p$. As shown in ref. 17, the connected part $\Psi_c(x, c)$ can be decomposed further into the lace parts $\Psi_l(x, n)$ such that

$$\hat{\Psi}_c(k, z) = \sum_{l=0}^{\infty} (-1)^l \, \hat{\Psi}_l(k, z) \tag{16}$$

for $0 < p \leqslant p_c$, $\mathrm{Re}(z) < m_p$, $k \in [-\pi, \pi]^d$. In order to analyze $\Psi_c(x, n)$ we introduce various necessary quantities and derive some basic estimates that will be needed for our analysis. Since these quantities are rather complicated to define, it is convenient to represent them in terms of Feynman diagrams. In the Feynman diagrams, the following notations will be used:

$(y, m) \to (x, n)$	represents	$\varphi(x-y, n-m)$
$(y, m) \not\twoheadrightarrow (x, n)$	represents	$\Psi(x-y, n-m)$
$(y, m) \to \| b$	represents	$P_p((y, m) \to \text{bottom of } b)\, P_p(b \text{ open})$

Further convention is that labeled vertices and bonds are fixed, but unlabeled vertices and bonds are summed over the lattice $Z^d \times Z$.

In Section 2, we introduce the bubble functions $Q^{(\alpha)}_{(y,m)}(x, n)$, $\alpha = 1, 2, 3$, the triangle functions $T^{(\alpha)}_{(y,m)}(x, n)$, $\alpha = 1, 2, 3$, and the funtions $E_l(x, n)$, $l \in \{0, 1, 2,...\}$; see Figs. 1–4 for the Feynman diagrams of some of these typical functions. We then use them to bound the lace parts as in the following lemma.

Lemma 1. For $l \in \{0, 1, 2,...\}$ we have

$$\Psi_l(x, n) \leqslant E_l(x, n) \tag{17}$$

The proof of (17) can be found in ref. 17. Furthermore, we show in Section 2 that the quantities $|\hat{E}_l(k, z)|$ and their derivatives can be estimated in terms of the Fourier–Laplace transforms of the bubble functions and the triangle functions for $k \in [-\pi, \pi]^d$ and $\mathrm{Re}(z) < m_p$.

In Section 3, we apply the basic estimates in Section 2 and the bootstrapping argument "(P_{2K}) implies (P_K)" to obtain estimates on $\hat{\Psi}_c(k, z)$ uniformly for $k \in [-\pi, \pi]^d$ and $\mathrm{Re}(z) \leqslant m_p$ and then use them to prove Theorem 2. Notice that in ref. 17 the bootstrapping argument was used to produce estimates on $\hat{\Psi}_c(k, z)$ for $\mathrm{Re}(z) \leqslant 0$. However, the proofs of Theorems 2 and 3 require similar estimates for a larger range of z values, $\mathrm{Re}(z) \leqslant m_p$; therefore, more careful analysis is needed in this paper.

We note that Campanino *et al.*[6] showed a Gaussian limit for the rescaled connectivity function of nearest-neighbor unoriented percolation for $p < p_c$ in all dimensions. We believe (even though we have not yet checked) that their method can be applied to establish a Gaussian limit for the rescaled connectivity function of finite-range oriented percolation for $p < p_c$ in all dimensions since the power expansion of $[\hat{\varphi}(0, z)]^{-1}$ (with respect to $w = e^z$) is expected to have a radius of convergence which is strictly larger than e^{m_p} in the subcritical region and from this a Gaussian limit can be shown. Notice that the method of Campanino *et al.* can produce a Gaussian limit in x space instead of a Gaussian limit in k space as stated in our Theorem 2. While the method of Campanino *et al.* seems more natural than the lace expansion method for the study of Gaussian fluctuation of the connectivity function in the subcritical regime, their method cannot be extended to the case at p_c since at this critical density it is expected that the power series of $\hat{\varphi}(0, z)$ and $[\hat{\varphi}(0, z)]^{-1}$ (expanded with respect to $w = e^z$) would have the same radius of convergence. Furthermore, the Gaussian scaling limit of the connectivity function at criticality is not expected to hold below the upper critical dimension. Thus we need a kind of Tauberian theorem to take care of this complication. To do this we use the fractional derivative method as shown in Lemmas 3.1–3.4 of ref. 13. The use of fractional derivatives in the lace expansion method has been very effective in showing the Gaussian scaling limit for self-avoiding random walks in high dimensions (including $d = 5, 6$).[13,14] Here we define the fractional derivative δ_z^ε of $f(z) \equiv \sum_{n=0}^\infty a_n e^{zn}$ as

$$\delta_z^\varepsilon f(z) = \sum_{n=1}^\infty n^\varepsilon a_n e^{zn} \tag{18}$$

We observe that (18) produces the usual derivatives for $\varepsilon = 1, 2, \ldots$. [Notice that in ref. 13 the fractional derivative δ_w^ε of $\tilde{f}(w) \equiv \sum_{n=0}^\infty a_n w^n$ is defined as

$$\delta_w^\varepsilon \tilde{f}(w) = \sum_{n=1}^\infty n^\varepsilon a_n w^n$$

Thus our definition of the fractional derivative is slightly different from the one defined in ref. 13; however, such a difference is inessential and it was introduced only for our notational convenience.]

To carry out this Tauberian argument we need the following lemma.

Lemma 2. Let $p \in (0, p_c]$ be given in the oriented percolation models mentioned in Theorem 2. Then for any positive number $\varepsilon < 1/2$ we can find constants $c_4(\varepsilon)$, $c_5(\varepsilon)$ such that

$$\sum_{x \in Z^d} \sum_{n=1}^\infty n^{1+\varepsilon} |\Psi_c(x, n)| e^{nm_p} \leqslant c_4(\varepsilon) \tag{19}$$

$$\sum_{x \in Z^d} \sum_{n=1}^\infty n^\varepsilon |x|^2 |\Psi_c(x, n)| e^{nm_p} \leqslant c_5(\varepsilon) \tag{20}$$

We discuss the proof of Lemma 2 in Section 4, and use this result to prove Theorem 3 in Section 5.

Remark 4. We need the fraction $\varepsilon < 1/2$ because it is expected that $\varphi(x, n)$ is comparable to the n-step transition function $G(x, n)$ of the corresponding random walk, and with this belief (19) and (20) may be infinite for $\varepsilon = 1$ in dimensions $d \leqslant 6$. Nevertheless, with sufficiently high dimensions we may use $\varepsilon = 1$ instead.

Remark 5. Recently, by using the lace expansion method and a sophisticated computer-assisted proof, Hara and Slade have been able to show a Gaussian limit for self-avoiding random walks in Z^d with $d > 4$.[13,14] It would be interesting to see whether such a technique can be applied to produce Gaussian scaling limits for oriented percolation right at p_c down to (possibly including) the upper critical dimension.

In the following sections, we refer only to the spread-out models for simplicity and we let the reader extend the ideas discussed there to the nearest-neighbor case.

2. FEYNMAN DIAGRAMS AND RELATED ESTIMATES

In this section we introduce various functions, represent them by Feynman-type diagrams with conventions mentioned in Section 1, and then

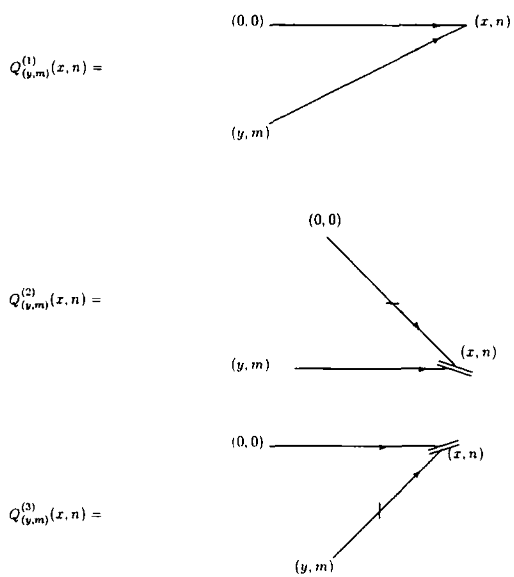

Fig. 1. Feynman diagrams of bubble functions.

use them to derive some estimates that will be needed for the proofs of our results. We define the bubble functions

$$Q^{(1)}_{(y,m)}(x,n)=\varphi(x,n)\,\varphi(x-y,n-m) \tag{21}$$

$$Q^{(2)}_{(y,m)}(x,n)=\Psi(x,n)\sum_{u\in Z^d}\varphi(u-y,n-1-m)\,p_{u,x} \tag{22}$$

$$Q^{(3)}_{(y,m)}=\sum_{u\in Z^d}\varphi(u,n-1)\,p_{u,x}\,\Psi(x-y,n-m) \tag{23}$$

and represent them by the diagram in Fig. 1.

We define the triangle $T[((x_1,n_1),(x_2,n_2)),(b,(u,n))]$ by

$$P_p((x_1,n_1)\to \text{bottom of } b)\,P_p(b \text{ open})\,P_p(\text{top of } b\to(u,n))\,\Psi(u-x_2,n-n_2)$$

then set

$$T[((x_1,n_1),(x_2,n_2)),((u,n),b)]=T[((x_2,n_2),(x_1,n_1)),(b,(u,n))]$$

and represent them by the diagrams in Fig. 2.

We also introduce the triangle functions

$$T^{(1)}_{(y,m)}(x,n)=\varphi(x,n)\sum_{(u',n')}\varphi(u'-y,n'-m)\,\varphi(x-u',n-n') \tag{24}$$

$$T^{(2)}_{(y,m)}(x,n)=\Psi(x,n)\sum_{(u',n')}\sum_{u\in Z^d}\varphi(u-y,n'-1-m)\,p_{u,u'}\,\varphi(x-u',n-n') \tag{25}$$

$$T^{(3)}_{(y,m)}(x,n)=\sum_{u\in Z^d}\varphi(u,n-1)\,p_{u,x}\sum_{(u',n')}\Psi(u'-y,n'-m)\,\varphi(x-u',n-n') \tag{26}$$

They are represented by the diagrams in Fig. 3.

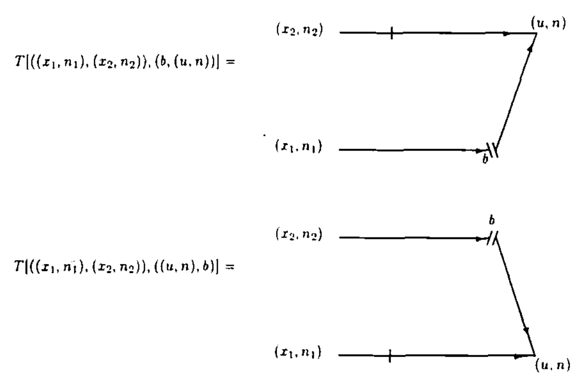

Fig. 2. Feynman diagrams of $T[(x_1,n_1),(x_2,n_2),(b,(u,n))]$ and $T[(x_1,n_1),(x_2,n_2),((u,n),b)]$.

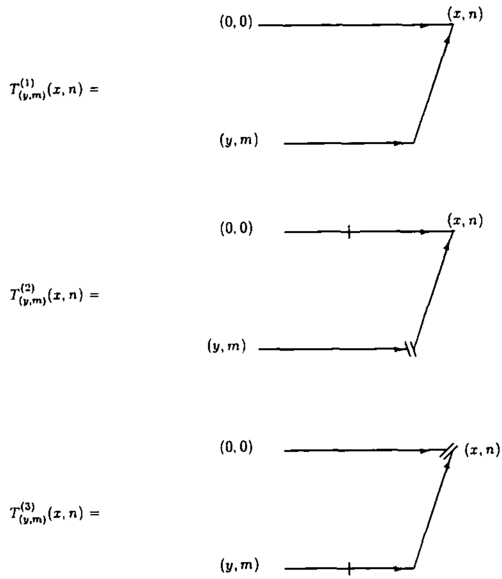

Fig. 3. Feynman diagrams of triangle functions.

Furthermore, given l pairs of site and bond $\{(b_i, (u_i, n_i)), i = 1, 2,..., l\}$, $l = 1, 2,...,$ we set $\Gamma_i(b_i, (u_i, n_i)) = (b_i, (u_i, n_i))$ or $((u_i, n_i), b_i)$, depending on whether Γ_i is the identity map or the permutation of site and bond. Let $\Gamma = (\Gamma_1,..., \Gamma_l)$. The diagram

$$E_l\{\Gamma, (y, m), b_i, (u_i, n_i), (x, n); i = 1, 2,..., l\}$$

is defined by

$$\begin{aligned} &T[((y, m)(y, m)), \Gamma_1(b_1, (u_1, n_1))] \\ &\quad \times \prod_{i = 2,...,l} T[\Gamma_{i-1}(b_{i-1}, (u_{i-1}, n_{i-1})), \Gamma_i(b_i, (u_i, n_i))] \\ &\quad \times P_p((u_l, n_l) \to (x, n))\, P_p((\text{top of } b_l \to (x, n)) \end{aligned} \tag{27}$$

For $l = 4$ and $\Gamma_1, \Gamma_3, \Gamma_4 = id$, $\Gamma_2 =$ permutation of bond b_2 and site (u_2, n_2), the diagram

$$E_l\{\Gamma, (y, m), b_i, (u_i, n_i), (x, n); i = 1, 2,..., l\}$$

is represented by the diagram in Fig. 4.

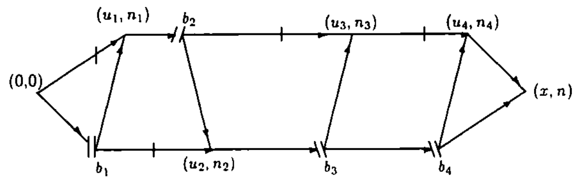

Fig. 4. Feynman diagram of $E_4\{\Gamma, (0,0), b_i, (x_i, n_i); i=1,2,3,4\}$.

The function $E_l(x, n)$ is defined by summing

$$E_l\{\Gamma, (0,0), b_i, (u_i, n_i), (x, n); i=1, 2,..., l\}$$

over the set $\{\Gamma, b_i, (u_i, n_i); i=1, 2,..., l\}$. For the sake of notation, we use $E_0(x, n)$ to denote $[\Psi(x, n)]^2$.

We then introduce the Fourier–Laplace transform

$$\hat{E}_l(k, z) = \sum_{(x,n)} e^{ik.x} e^{zn} E_l(x, n) \tag{28}$$

for $k \in [-\pi, \pi]^d$, $\text{Re}(z) < m_p$, and define the Fourier–Laplace transforms $\hat{Q}^{(\alpha)}_{(y,m)}(k, z)$, $\hat{T}^{(\alpha)}_{(y,m)}(k, z)$ of the functions $Q^{(\alpha)}_{(y,m)}(x, n)$, $T^{(\alpha)}_{(y,m)}(x, n)$ for each $\alpha = 1, 2, 3$ similarly.

Now we want to derive some estimates concerning $\hat{E}_l$, $\hat{Q}^{(\alpha)}_{(y,m)}$, $\hat{T}^{(\alpha)}_{(y,m)}$, and their derivatives. It is easy to see that

$$E_0(x, n) = [\Psi(x, n)]^2 \leqslant Q^{(2)}_{(0,0)}(x, n) \leqslant T^{(2)}_{(0,0)}(x, n) \tag{29}$$

Hence,

$$\hat{E}_0(0, s) \leqslant \hat{Q}^{(2)}_{(0,0)}(0, s) \leqslant \hat{T}^{(2)}_{(0,0)}(0, s) \tag{30}$$

In estimating $\hat{E}_l$, $l \geqslant 1$, and their derivatives we need to use repeatedly the inequality

$$\left| \sum_{(x,n)} f_1(x, n) f_2(x, n) \right| \leqslant \{\sup_{(x,n)} |f_1(x, n)|\} \sum_{(x,n)} |f_2(x, n)| \tag{31}$$

Inserting (27) in the summation of $\hat{E}_l(k, z)$ and using (31) repeatedly in summing over the set $\{\Gamma, b_i, (u_i, n_i); i=1,..., l, (x, n)\}$ in the order from right to left, we obtain, for $s < m_p$ and $l \geqslant 1$,

$$\hat{E}_l(0, s) \leqslant 2^l [\sup_{(y,m); \alpha=2,3} \hat{T}^{(\alpha)}_{(y,m)}(0, s)]^l \sup_{(y,m)} \hat{Q}^{1)}_{(y,m)}(0, s) \tag{32}$$

Each derivative of $\hat{E}_l(k, z)$ evaluated at $k=0$ and $z=s<m_p$ can be estimated in the same way with the only change due to the factor that corresponds to the derivative. To estimate $\partial_z \hat{E}_l(k, z)$, we proceed as follows. We let $(v_0, m_0)=(0, 0)$, $(v_{l+1}, m_{l+1})=(x, n)$, and set $\{(v_i, m_i); i=1,..., l\}$ the l corners along the top route from $(0, 0)$ to (x, n) of the diagram

$$E_l\{\Gamma, (0, 0), b_i, (u_i, n_i), (x, n); i=1, 2,..., l\}$$

We then use $n=\sum_{i=0}^{l} (m_{i+1}-m_i)$ and distribute each factor $(m_{j+1}-m_j)$ in the corresponding diagram from (v_j, m_j) to (v_{j+1}, m_{j+1}). Using the same technique as in Section 3.2 of ref. 11, we estimate the contribution of the term corresponding to the factor $(m_{j+1}-m_j)$ and then sum over j from 0 to l to obtain, for $s<m_p$ and $l\geqslant 1$,

$$\begin{aligned}\partial_z \hat{E}_l(0, s) \leqslant 2^l [\sup_{(y,m);\alpha=2,3} \hat{T}^{(\alpha)}_{(y,m)}(0, s)]^l \sup_{(y,m)} \partial_z \hat{Q}^{(1)}_{(y,m)}(0, s)\\ +2^l l [\sup_{(y,m);\alpha=2,3} \hat{T}^{(\alpha)}_{(y,m)}(0, s)]^{l-1}\\ \times [\sup_{(y,m);\alpha=2,3} \partial_z \hat{Q}^{(\alpha)}_{(y,m)}(0, s)] \sup_{(y,m)} \hat{T}^{(1)}_{(y,m)}(0, s) \qquad (33)\end{aligned}$$

To bound the second derivative $\partial_{\mu\mu} \hat{E}_l(k, z)$ evaluated at $k=0$, $z=s<m_p$, we use the inequality

$$|x|^2 \leqslant (l+1) \sum_{i=0}^{l} |v_{i+1}-v_i|^2$$

and apply the same argument. We then have

$$\begin{aligned}\partial_{\mu\mu} \hat{E}_l(0, s) \leqslant 2^l (l+1) [\sup_{(y,m);\alpha=2,3} \hat{T}^{(\alpha)}_{(y,m)}(0, s)]^l \sup_{(y,m)} \partial_{\mu\mu} \hat{Q}^{(1)}_{(y,m)}(0, s)\\ +2^l l (l+1) [\sup_{(y,m);\alpha=2,3} \hat{T}^{(\alpha)}_{(y,m)}(0, s)]^{l-1}\\ \times [\sup_{(y,m);\alpha=2,3} \partial_{\mu\mu} \hat{Q}^{(\alpha)}_{(y,m)}(0, s)] \sup_{(y,m)} \hat{T}^{(1)}_{(y,m)}(0, s) \qquad (34)\end{aligned}$$

Ass we will see later, the above estimates will be needed in proving Theorem 2. For proving Lemma 2, in addition to these estimates, we also need to bound $\partial_z \partial_{\mu\mu} \hat{E}_l(0, s)$ and $\partial_{zz} \hat{E}_l(0, s)$. To do this we need to distribute the factors such as n and $|x|^2$ along the top as well as the bottom paths of the diagram. We let $(v'_0, m'_0)=(0, 0)$, $(v'_{l+1}, m'_{l+1})=(x, n)$, and set

$\{(v_i', m_i'); i=1,..., l\}$ the l corners along the lower route from (0, 0) to (x, n) of the diagram

$$E_l\{\Gamma, (0, 0), b_i, (u_i, n_i), (x, n); i=1, 2,..., l\}$$

We then use

$$n\,|x|^2 \leqslant (l+1) \sum_{i, j \in \{0, 1,..., l\}} (m_{i+1} - m_i)\,|v_{j+1}' - v_j'|^2$$

and bound $\partial_z \partial_{\mu\mu} \hat{E}_l(0, s)$ above by the sum of five terms (I), (II), (III), (IV), and (V) corresponding to $\{i=j=l\}$, $\{i=j\neq l\}$, $\{i\neq l, j=l\}$, $\{j\neq l, i=l\}$, and $\{i\neq j, i\neq l, j\neq l\}$, respectively, as follows:

$$\partial_z \partial_{\mu\mu} \hat{E}_l(0, s) \leqslant (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}) \tag{35}$$

with

$$(\mathrm{I}) \leqslant \partial_z \partial_{\mu\mu} \hat{E}_l(0, s) \leqslant 2^l(l+1)\,[\sup_{(y,m);\,\alpha=2,3} \hat{T}^{(\alpha)}_{(y,m)}(0, s)]^l \sup_{\alpha=1,2,3} \tilde{Q}^{(\alpha)}_{z\mu\mu}(0, s)$$

$$(\mathrm{II}) \leqslant 2^l l(l+1)\,[\sup_{(y,m);\,\alpha=2,3} \hat{T}^{(\alpha)}_{(y,m)}(0, s)]^{l-1} \sup_{\alpha=1,2,3} \tilde{Q}^{(\alpha)}_{z\mu\mu}(0, s) \times \sup_{(y,m)} \hat{T}^{(1)}_{(y,m)}(0, s)$$

$$(\mathrm{III}) \leqslant 2^l l(l+1)\,[\sup_{(y,m);\,\alpha=2,3} \hat{T}^{(\alpha)}_{(y,m)}(0, s)]^{l-1}\,[\sup_{(y,m);\,\alpha=2,3} \partial_z \hat{T}^{(\alpha)}_{(y,m)}(0, s)] \times \sup_{(y,m)} \partial_{\mu\mu} \hat{Q}^{(1)}_{(y,m)}(0, s)$$

$$(\mathrm{IV}) \leqslant 2^l l(l+1)\,[\sup_{(y,m);\,\alpha=2,3} \hat{T}^{(\alpha)}_{(y,m)}(0, s)]^{l-1}\,[\sup_{(y,m);\,\alpha=2,3} \partial_{\mu\mu} \hat{T}^{(\alpha)}_{(y,m)}(0, s)] \times \sup_{(y,m)} \partial_z \hat{Q}^{(1)}_{(y,m)}(0, s)$$

$$(\mathrm{V}) \leqslant 2^l(l+1)\,l(l-1)\,[\sup_{(y,m);\,\alpha=2,3} \hat{T}^{(\alpha)}_{(y,m)}(0, s)]^{l-2} \times [\sup_{(y,m);\,\alpha=2,3} \partial_{\mu\mu} \hat{Q}^{(\alpha)}_{(y,m)}(0, s)] \times [\sup_{(y,m);\,\alpha=2,3} \partial_z \hat{T}^{(\alpha)}_{(y,m)}(0, s)] \sup_{(y,m)} \hat{T}^{(1)}_{(y,m)}(0, s)$$

where

$$\tilde{Q}^{(\alpha)}_{z\mu\mu}(0, s) = \sup_{(y,m)} \sum_{(x,n)} e^{ns} n\,|x-y|^2\, Q^{(\alpha)}_{(y,m)}(x, n)$$

Similarly, to bound $\partial_{zz}\hat{E}_l(0, s)$ we use

$$n^2=(l+1)\sum_{i,j\in\{0,1,\dots,l\}}(m_{i+1}-m_i)(m'_{j+1}+m'_j)$$

and apply the same argument. Note that in this case the terms corresponding to $\{i\neq l,\ j=l\}$ and $\{j\neq l,\ i=l\}$ are equal. We then have

$$\begin{aligned}
&\partial_{zz}\hat{E}_l(0, s)\\
&\quad\leqslant 2^l\Big[\sup_{(y,m);\alpha=2,3}\hat{T}^{(\alpha)}_{(y,m)}(0,s)\Big]^l\sup_{\alpha=1,2,3}\tilde{Q}^{(\alpha)}_{zz}(0,s)\\
&\qquad+2^l l\Big[\sup_{(y,m);\alpha=2,3}\hat{T}^{(\alpha)}_{(y,m)}(0,s)\Big]^{l-1}\sup_{\alpha=1,2,3}\tilde{Q}^{(\alpha)}_{zz}(0,s)\sup_{(y,m)}\hat{T}^{(1)}_{(y,m)}(0,s)\\
&\qquad+2_x 2^l l\Big[\sup_{(y,m);\alpha=2,3}\hat{T}^{(\alpha)}_{(y,m)}(0,s)\Big]^{l-1}\\
&\qquad\times\Big[\sup_{(y,m);\alpha=2,3}\partial_z\hat{T}^{(\alpha)}_{(y,m)}(0,s)\Big]\sup_{(y,m)}\partial_z\hat{Q}^{(1)}_{(y,m)}(0,s)\\
&\qquad+2^l l(l-1)\Big[\sup_{(y,m);\alpha=2,3}\hat{T}^{(\alpha)}_{(y,m)}(0,s)\Big]^{l-2}\Big[\sup_{(y,m);\alpha=2,3}\partial_z\hat{Q}^{(\alpha)}_{(y,m)}(0,s)\Big]\\
&\qquad\times\Big[\sup_{(y,m);\alpha=2,3}\partial_z\hat{T}^{(\alpha)}_{(y,m)}(0,s)\Big]\sup_{(y,m)}\hat{T}^{(1)}_{(y,m)}(0,s)
\end{aligned}\tag{36}$$

where

$$\tilde{Q}^{(\alpha)}_{zz}(0,s)=\sup_{(y,m)}\sum_{(x,n)}e^{ns}\,|n|\cdot|n-m|\,Q^{(\alpha)}_{(y,m)}(x,n)$$

Next we want to bound the bubble functions, the triangle functions, and their derivatives in terms of $\hat{\varphi}(k, z)$ and its derivatives. The following lemmas can be obtained by applying the Hausdorff–Young inequality. Notice that the integral $\iint f(k, t)\,dk\,dt$ means

$$(2\pi)^{-d-1}\iint_{(k,t)\in[-\pi,\pi]^d\times[-\pi,\pi]}f(k,t)\,dk\,dt$$

and the derivative ∂_a can be replaced by ∂_z, $\partial_{\mu\mu}$, ∂_{zz}, $\partial_{z\mu\mu}$, or ∂_0 (which is defined as the identity map).

Lemma 3.

$$\sup_{(y,m)}\partial_a\hat{Q}^{(1)}_{(y,m)}(0,s)\leqslant\iint dk\,dt\,|\partial_a\hat{\varphi}(k,s+it)\,\hat{\varphi}(k,it)|\tag{37}$$

$$\sup_{(y,m)}\partial_a\hat{Q}^{(2)}_{(y,m)}(0,s)\leqslant\iint dk\,dt\,|\partial_a[\hat{\varphi}(k,s+it)]\,\hat{\varphi}(k,it)\,\hat{p}(k)|\tag{38}$$

$$\sup_{(y,m)}\partial_a\hat{Q}^{(3)}_{(y,m)}(0,s)\leqslant\iint dk\,dt\,|\partial_a[\hat{\varphi}(k,s+it)\,\hat{p}(k)]\,\hat{\varphi}(k,it)|\tag{39}$$

Lemma 4

$$\sup_{(y,m)} \partial_a \hat{T}^{(1)}_{(y,m)}(0,s) \leqslant \iint dk\, dt\, |\partial_a \hat{\varphi}(k, s+it)[\hat{\varphi}(k, it)]^2| \tag{40}$$

$$\sup_{(y,m)} \partial_a \hat{T}^{(2)}_{(y,m)}(0,s) \leqslant \iint dk\, dt\, |\partial_a [\hat{\varphi}(k, s+it)][\hat{\varphi}(k, it)]^2\, \hat{p}(k)| \tag{41}$$

$$\sup_{(y,m)} \partial_a \hat{T}^{(3)}_{(y,m)}(0,s) \leqslant \iint dk\, dt\, |\partial_a [\hat{\varphi}(k, s+it)\, \hat{p}(k)]| \cdot |\hat{\varphi}(k, it)|^2 \tag{42}$$

Lemma 5

$$\tilde{Q}^{(1)}_{z\mu\mu}(0,s) \leqslant \iint dk\, dt\, |\partial_z \hat{\varphi}(k, s+it)\, \partial_{\mu\mu} \hat{\varphi}(k, -it)| \tag{43}$$

$$\tilde{Q}^{(2)}_{z\mu\mu}(0,s) \leqslant 2 \iint dk\, dt\, |\partial_z \hat{\varphi}(k, s+it)| \times \{|\partial_{\mu\mu} \hat{\varphi}(k, -it)| \cdot |\hat{p}(k)| + |\hat{\varphi}(k, -it)| \cdot |\partial_{\mu\mu} \hat{p}(k)|\} \tag{44}$$

$$\tilde{Q}^{(3)}_{z\mu\mu}(0,s) \leqslant \iint dk\, dt\, \{|\partial_z \hat{\varphi}(k, s+it)| + |\hat{\varphi}(k, s+it)|\} \times |\hat{p}(k)| \cdot |\partial_{\mu\mu} \hat{\varphi}(k, -it)| \tag{45}$$

$$\tilde{Q}^{(1)}_{zz}(0,s) \leqslant \iint dk\, dt\, |\partial_z \hat{\varphi}(k, s+it)\, \partial_z \hat{\varphi}(k, -it)| \tag{46}$$

$$\tilde{Q}^{(2)}_{zz}(0,s) \leqslant \iint dk\, dt\, |\partial_z \hat{\varphi}(k, s+it)| \times \{|\partial_z \hat{\varphi}(k, -it)| + |\hat{\varphi}(k, -it)|\}\, |\hat{p}(t)| \tag{47}$$

By symmetry, $\tilde{Q}^{(3)}_{zz}(0,s)$ is bounded by the right-hand side of (47).

3. INFRARED BOUND

In this section we show the infrared bound (11) for the spread-out oriented percolation models in $Z^d \times Z$, $d > 4$, as mentioned in Theorem 2.

First, we point out that there exist δ, δ_1 with $0 < \delta \leqslant \delta_1$ such that

$$|\hat{D}_L(k)| \leqslant 2/3 \qquad \text{if} \quad |k| \geqslant \delta \tag{48}$$

$$|1 - \hat{D}_L(k)| \geqslant \text{const} \cdot L^2\, |k|^2 \qquad \text{if} \quad |k| \leqslant \delta_1 \tag{49}$$

for sufficiently large L. In fact, it was shown in Lemma 5.5 of ref. 11 that this can be done with δ and δ_1 chosen as follows. Set $\delta = 3\pi(2L)^{-1} \int_{x\in R^d} |\partial_1 g(x)|\, dx$. Since g is continuous at 0 and $g(0)>0$, we can choose $M>0$ so small that $(3/2)\int_{x\in R^d} |\partial_1 g(x)|\, dx < 1/M$ and $\int_{\{x\in R^d: |x|\leq M\}} g(x)\, dx > 0$. Choose $\delta_1 = \pi(LM)^{-1}$; then $\delta_1 \geq \delta$ and (48) and (49) hold.

Second, we only need to show the infrared bound (11) for $p\in[p_L, p_c)$, since it can be shown easily from inequality (5) for $p\in(0, p_L]$. Thus Theorem 2 will follow from the renewal equation (15) if we can show that for every $p\in[p_L, p_c]$,

$$\sum_{(x,n)} |\Psi_c(x,n)|\, e^{nm_p} \leq \varepsilon(L) \qquad \text{with} \quad \varepsilon(L)\to 0 \quad \text{as} \quad L\to\infty \tag{50}$$

and

$$|F(k, m_p - s + it)| \geq \tfrac{1}{2} pp_L^{-1} e^{m_p} |1 - e^{-s+it}\hat{D}_L(k)| \tag{51}$$

where

$$F(k, m_p - s + it) \equiv 1 - pp_L^{-1}\hat{D}_L(k)\, e^{m_p - s + it}[1 + \hat{\Psi}_c(k, m_p - s + it)] \tag{52}$$

for every $(k, t)\in[-\pi, \pi]^d\times[-\pi, \pi]$ and $s\in(0, m_p + \log p - \log p_L + 1]$ (notice that in this case both m_p and $\log p - \log p_L$ are nonnegative, so $m_p + \log p - \log p_L \geq 1$, hence $(0, 1]\subset(0, m_p + \log p - \log p_L + 1]$).

In addition, the condition (51) can be replaced by

$$\tfrac{1}{2}|1 - e^{-s+it}\hat{D}_L(k)| \geq |\hat{\Psi}_c(0, m_p) - \hat{\Psi}_c(k, m_p - s + it)\, e^{-s+it}\hat{D}_L(k)| \tag{53}$$

as we can see in the following lemma.

Lemma 6. Assume that (50) and (53) hold for every $s\in(0, m_p + \log p - \log p_L + 1]$, $p\in(0, p_c]$, and $(k, t)\in[-\pi, \pi]^d\times[-\pi, \pi]$. Then (51) holds.

Proof. We observe from (50) that $\hat{\Psi}_c(0, m_p)$ is well defined, and also

$$\hat{\Psi}_c(0, m_p) = \lim_{s\downarrow 0} \hat{\Psi}_c(0, m_p - s) \tag{54}$$

by the dominated convergence theorem. Furthermore, it is easy to see from (13) that

$$\lim_{s\downarrow 0} \hat{\varphi}(0, m_p - s) = \infty \tag{55}$$

Since

$$1 - pp_L^{-1} e^{m_p - s}[1 + \hat{\Psi}_c(0, m_p - s)] = \frac{1 + \hat{\Psi}_c(0, m_p - s)}{\hat{\varphi}(0, m_p - s)} \tag{56}$$

[this follows easily from the renewal equation (15)] we have by (54) and (55)

$$1 - pp_L^{-1} e^{m_p}[1 + \hat{\Psi}_c(0, m_p)] = \lim_{s \downarrow 0} \frac{1 + \hat{\Psi}_c(0, m_p - s)}{\hat{\varphi}(0, m_p - s)} = 0 \tag{57}$$

Since

$$\begin{aligned} F(k, m_p - s + it) & \\ &= 1 - pp_L^{-1} e^{m_p - s + it} \hat{D}_L(k)[1 + \hat{\Psi}_c(k, m_p - s + it)] \\ &= \{1 - pp_L^{-1} e^{m_p}[1 + \hat{\Psi}_c(0, m_p)]\} + pp_L^{-1} e^{m_p}[1 - e^{-s+it} \hat{D}_L(k)] \\ &\quad + pp_L^{-1} e^{m_p}[\hat{\Psi}_c(0, m_p) - \hat{\Psi}_c(k, m_p - s + it) e^{-s+it} \hat{D}_L(k)] \end{aligned} \tag{58}$$

then by (57) and the simple triangular inequality

$$\begin{aligned} |F(k, m_p - s + it)| & \\ &\geqslant pp_L^{-1} e^{m_p} |1 - e^{-s+it} \hat{D}_L(k)| \\ &\quad - pp_L^{-1} e^{m_p} |\hat{\Psi}_c(0, m_p) - \hat{\Psi}_c(k, m_p - s + it)\, e^{-s+it} \hat{D}_L(k)| \end{aligned} \tag{59}$$

Then (51) follows from (59) and (53).

The next lemma shows that (50) and (53) can be replaced further by the following conditions:

$$\sum_{(x,n)} n\, |\Psi_c(x, n)|\, e^{nm_p} \leqslant \varepsilon_z(L) \tag{60}$$

$$\sum_{(x,n)} |x|^2\, |\Psi_c(x, n)|\, e^{nm_p} \leqslant \varepsilon_{\mu\mu}(L) \tag{61}$$

where $p \in (0, p_c]$ and $\varepsilon_z(L)$, $\varepsilon_{\mu\mu}(L) \to 0$ as $L \to \infty$.

Lemma 7. Assume that (60) and (61) hold. Then (50) and (53) follow.

Proof. Clearly (50) follows from (60). To show (53), we consider two cases: $|k| \geqslant \delta$ and $|k| \leqslant \delta_1$, where δ and δ_1 are chosen as in (48) and (49). For large k, $|k| \geqslant \delta$, and sufficiently large L the inequality (53) follows from (50). In any case we have the lower bounds

$$\begin{aligned} |1 - e^{-s+it} \hat{D}_L(k)|^2 &= (1 - e^{-s})^2 + e^{-2s}\, |1 - e^{it} \hat{D}_L(k)|^2 \\ &\quad + 2e^{-s}(1 - e^{-s})[1 - \hat{D}_L(k) \cos t] \\ &\geqslant (1 - e^{-s})^2 + e^{-2s}\, |1 - e^{it} \hat{D}_L(k)|^2 \end{aligned} \tag{62}$$

$$|1 - e^{it} \hat{D}_L(k)|^2 \geqslant \begin{cases} \frac{1}{16}\, |1 - \hat{D}_L(k)|^2 + (1 - \cos t) & \text{if } \hat{D}_L(k) \geqslant 0 \\ \frac{1}{16}\, |1 + \hat{D}_L(k)|^2 + (1 + \cos t) & \text{if } \hat{D}_L(k) \leqslant 0 \end{cases} \tag{63}$$

For small k, $|k| \leqslant \delta_1$, and $\hat{D}_L(k) \leqslant 0$ we use (63) to obtain

$$\begin{aligned} 1+\hat{D}_L(k) &= \sum_x L^{-d} g(x/L)[1+\cos(k\cdot x)] \\ &\geqslant \sum_{x:|k\cdot x| \leqslant \pi/2} L^{-d} g(x/L) \\ &\geqslant \sum_{x:|x| \leqslant LM/2} L^{-d} g(x/L) \\ &\sim \int_{\{x:\,|x| \leqslant M/2\}} g(x)\,dx > 0 \end{aligned} \tag{64}$$

By (62) the left-hand side of (53) is bounded by a constant; hence, (53) follows for sufficiently large L from (50) and (64).

For the remaining case, $|k| \leqslant \delta_1$ and $\hat{D}_L(k) \geqslant 0$, we observe from (49), (62), and (63) that

$$|1-e^{-s+it}\hat{D}_L(k)| \geqslant c_4 L^2\,|k|^2 + c_5\,|t| + c_6\,|s| \tag{65}$$

Therefore in order to obtain (53) for large enough L it is enough to show

$$|\hat{\Psi}_c(0,m_p) - \hat{\Psi}_c(k,m_p+z)\,e^z\hat{D}_L(k)| = o(L)\,L^2\,|k|^2 + o(L)\,|t| + o(L)\,|s| \tag{66}$$

To show (66) we proceed as follows. We have, for $z=-s+it$

$$\begin{aligned} &|\hat{\Psi}_c(0,m_p) - \hat{\Psi}_c(k,m_p+z)\,e^z\hat{D}_L(k)| \\ &\quad\leqslant |\hat{\Psi}_c(0,m_p) - \hat{\Psi}_c(k,m_p)| + |\hat{\Psi}_c(k,m_p)(1-e^z)| \\ &\qquad + |\hat{\Psi}_c(k,m_p)\,e^z[1-\hat{D}_L(k)]| + |\hat{\Psi}_c(k,m_p) - \hat{\Psi}_c(k,m_p-z)|\cdot|e^z\hat{D}_L(k)| \end{aligned}$$

The first term is equal to

$$\begin{aligned} &\left|\int_0^1 dr\,(1-r)\frac{d^2}{dr^2}\hat{\Psi}_c(rk,m_p)\right| \\ &\quad\leqslant 1/2 \sup_{0\leqslant r\leqslant 1}\left|\sum_{\mu\nu=1}^d k_\mu k_\nu\,\partial_{\mu\nu}\hat{\Psi}_c(rk,m_p)\right| \\ &\quad\leqslant 1/2 \sum_{(x,n)}\sum_{\mu\nu=1}^d |k_\mu|\cdot|k_\nu|\cdot|x_{(\mu)}|\cdot|x_{(\nu)}|\cdot|\Psi_c(x,n)|\,e^{nm_p} \end{aligned} \tag{67}$$

where $x_{(\mu)}$, $\mu = 1,..., d$ are the coordinates of x. But

$$\sum_{\mu\nu=1}^{d} |k_\mu| \cdot |k_\nu| \cdot |x_{(\mu)}| \cdot |x_{(\nu)}| \leqslant \tfrac{1}{4} \sum_{\mu\nu=1}^{d} (|k_\mu|^2 + |k_\nu|^2)(|x_{(\mu)}|^2 + |x_{(\nu)}|^2)$$

$$\leqslant \tfrac{1}{2} \sum_{\mu\nu=1}^{d} (|k_\mu|^2 + |k_\nu|^2) \, |x|^2 = d \, |k|^2 \, |x|^2$$

so the right-hand side of (67) is bounded above by

$$\tfrac{1}{2} d \, |k|^2 \sum_{(x,n)} |x|^2 \, |\Psi_c(x, n)| \, e^{nm_p} \leqslant \tfrac{1}{2} d |k|^2 \, \varepsilon_{\mu\mu}(L)$$

The second term is bounded above by const $\cdot \, \varepsilon(L)(|s| + |t|)$, the third term is bounded above by $\varepsilon(L)$ const $\cdot \, L^2 \, |k|^2$, and the fourth is bounded by $\varepsilon_z(L)$ const $\cdot \, (|t| + |s|)$. Combining the estimates of these four terms, we obtain (66). This shows Lemma 7.

Thus to show Theorem 2 reduces to showing (60) and (61). To do this we apply the bootstrapping argument which has been successfully used for studying various models arising in statistical physics; see, for instance, refs. 5, 19, and 11–13. The argument given below follows the same idea as the one discussed in ref. 11.

It is easy to see from (58), (57), and (53) that

$$2pp_L^{-1} e^{m_p} \, |1 - e^{-s+it} \hat{D}_L(k)| \geqslant |F(k, m_p - s + it)| \tag{68}$$

This suggests that $\Psi(x, n)$ is comparable to the n-step transition function $G(x, n)$ of the corresponding random walk. This observation leads us to further results as follows.

Lemma 8. Let $0 < p < p_c$ and $\tilde{m} < m_p$ be such that the following inequalities hold:

$$e^{-1} \leqslant pp_L^{-1} e^{\tilde{m}} \tag{69}$$

$$\sum_{(x,n)} |\Psi_c(x, n)| \, e^{n\tilde{m}} \leqslant \tilde{\varepsilon}(L) \tag{70}$$

$$\sum_{(x,n)} n \, \Psi_c(x, n)| \, e^{n\tilde{m}} \leqslant \tilde{\varepsilon}_z(L) \tag{71}$$

$$\sum_{(x,n)} |x|^2 \, \Psi_c(x, n)| \, e^{n\tilde{m}} \leqslant \tilde{\varepsilon}_{\mu\mu}(L) \tag{72}$$

where $\tilde{\varepsilon}(L)$, $\tilde{\varepsilon}_z(L)$, $\tilde{\varepsilon}_{\mu\mu}(L) \to 0$ uniformly in p and $\tilde{m}$ as $L \to \infty$. Then for sufficiently large L we have

$$|F(k, \tilde{m} + it)| \geq \tfrac{1}{2} e^{\tilde{m}} \, |1 - e^{it} \hat{D}_L(k)| \tag{73}$$

and moreover, for every $(y, m) \in Z^d \times Z$, $\alpha' \in \{2, 3\}$, and $\alpha \in \{1, 2, 3\}$ we have

$$\hat{T}^{(1)}_{(y,m)}(0, \tilde{m}) \leq K_0 \tag{74}$$

$$\hat{Q}^{(1)}_{(y,m)}(0, \tilde{m}) \leq K_1 \tag{75}$$

$$\hat{T}^{(\alpha')}_{(y,m)}(0, \tilde{m}) \leq K_2 \iint dk\, dt\, |\hat{D}_L(k)| \cdot |1 - e^{it} \hat{D}_L(k)|^{-3} \tag{76}$$

$$\partial_z \hat{Q}^{(\alpha)}_{(y,m)}(0, \tilde{m}) \leq K_z \iint dk\, dt\, |\hat{D}_L(k)| \cdot |1 - e^{it} \hat{D}_L(k)|^{-3} \tag{77}$$

$$\partial_{\mu\mu} \hat{Q}^{(\alpha)}_{(y,m)}(0, \tilde{m}) \leq K_{\mu\mu} L^2 \iint dk\, dt\, |k|^2 \, |1 - e^{it} \hat{D}_L(k)|^{-4} \tag{78}$$

Notice that all of the integrals on the right-hand sides are finite (since $d > 4$) and $O(L^{-d} \log L)$,[17] and furthermore, the constants K are independent of p and $\tilde{m}$.

Proof. To prove (73) we just simply replace m_p, s in the proof of Lemma 7 by $\tilde{m}$, 0 and follow the same argument. For the remaining inequalities we only show (78) with $\alpha = 3$, which is the most difficult one, since the proofs of the others are similar. By Lemma 3,

$$\sup_{(y,m)} \partial_{\mu\mu} \hat{Q}^{(3)}_{(y,m)}(0, \tilde{m})$$

$$\leq \iint dk\, dt\, \{ |\partial_{\mu\mu} \hat{\varphi}(k, \tilde{m} + it)| + p p_L^{-1} \, |\partial_{\mu\mu} \hat{D}_L(k)| \} \, |\hat{\varphi}(k, it)| \tag{79}$$

But from the renewal equation (15) we have

$$\begin{aligned} \partial_{\mu\mu} \hat{\varphi}(k, z) = {} & F^{-1} \partial_{\mu\mu} \hat{\Psi}_c(k, z) - 2F^{-2} (\partial_\mu \hat{\Psi}_c(k, z)) \, \partial_\mu F \\ & - F^{-2} \hat{\Psi}_c(k, z) \, \partial_{\mu\mu} F + 2F^{-3} \hat{\Psi}_c(k, z) (\partial_\mu F)^2 \end{aligned} \tag{80}$$

Using (70)–(73), we can check for $z = \tilde{m} + it$ that

$$F^{-1} \, \partial_{\mu\mu} \hat{\Psi}_c(k, z) = (p p_L^{-1} e^{\tilde{m}})^{-1} \, |k|^2 \, O(L^2) \, |1 - e^{it} \hat{D}_L(k)|^{-1} \tag{81}$$

$$F^{-2} (\partial_\mu \hat{\Psi}_c(k, z)) \, \partial_\mu F = (p p_L^{-1} e^{\tilde{m}})^{-1} \, |k|^2 \, O(L^2) \, |1 - e^{it} \hat{D}_L(k)|^{-2} \tag{82}$$

$$F^{-2} \hat{\Psi}_c(k, z) \, \partial_{\mu\mu} F = (p p_L^{-1} e^{\tilde{m}})^{-1} \, |k|^2 \, O(L^2) \, |1 - e^{it} \hat{D}_L(k)|^{-2} \tag{83}$$

$$F^{-3} \hat{\Psi}_c(k, z) (\partial_\mu F)^2 = (p p_L^{-1} e^{\tilde{m}})^{-1} \, |k|^2 \, O(L^2) \, |1 - e^{it} \hat{D}_L(k)|^{-3} \tag{84}$$

Thus the right-hand side of (79) is bounded above by

$$\text{const}\cdot(pp_L^{-1}e^{\tilde m})^{-1}\iint dk\,dt(L^2|k|^2+pp_L^{-1}e^{\tilde m}L^2|k|^2)\sum_{j=1}^{3}|1-e^{it}\hat D_L(k)|^{-j-1}$$

which is bounded by the right-hand side of (78) for sufficiently large L by (69). This shows (78).

Let (P_{aK}) be the assumption that (74)–(78) hold with constants aK_0, aK_1, aK_2, aK_z, $aK_{\mu\mu}$ replacing K_0, K_1, K_2, K_z, $K_{\mu\mu}$ of the previous lemma. Since $\Psi(x,n)\leqslant(pp_L^{-1})^n G(x,n)$, without loss of generality we may assume (P_K) whenever (69) does not hold. From now on we fix these constants K. Next we want to show that Lemma 8 has its converse as in the following lemma.

Lemma 9. Let $p\in(0,p_c)$ and $\tilde m<m_p$ be such that (P_{2K}) holds. Then (70)–(72) follow.

Proof. Let $\bar\varepsilon(L)$, $\bar\varepsilon_z(L)$, $\bar\varepsilon_{\mu\mu}(L)$ be the quantities in the right-hand sides of (76)–(78) and notice that they are $O(L^{-d}\log L)$. Thus for $p\in(0,p_c)$ and $\tilde m<m_p$,

$$\sum_{(x,n)}|\Psi_c(x,n)|\,e^{n\tilde m}\leqslant\sum_{l=0}^{\infty}\hat\Psi_l(0,\tilde m)\leqslant\sum_{l=0}^{\infty}\hat E_l(0,\tilde m)$$

By (30) and (32) this is bounded above by $\bar\varepsilon(L)+\sum_{l=1}^{\infty}2^l\,2K_1$, which converges to 0 as $L\to\infty$. Similarly,

$$\sum_{(x,n)}n\,|\Psi_c(x,n)|\,e^{n\tilde m}\leqslant\sum_{l=0}^{\infty}\partial_z\hat\Psi_l(0,\tilde m)\leqslant\sum_{l=0}^{\infty}\partial_z\hat E_l(0,\tilde m)$$

By (33) this is bounded by

$$\bar\varepsilon_z(L)+\sum_{l=1}^{\infty}2^l\bar\varepsilon(L)^l\,\bar\varepsilon_z(L)+\sum_{l=1}^{\infty}l2^l\bar\varepsilon(L)^{l-1}\,\bar\varepsilon_z(L)\,2K_0$$

which converges to 0 as $L\to\infty$. Furthermore,

$$\sum_{(x,n)}|\Psi_c(x,n)|e^{n\tilde m}|x|^2\leqslant\sum_{l=0}^{\infty}\partial_{\mu\mu}\hat\Psi_l(0,\tilde m)\leqslant\sum_{l=0}^{\infty}\partial_{\mu\mu}\hat E_l(0,\tilde m)$$

By (34) this is bounded by

$$\bar{\varepsilon}_{\mu\mu}(L) + \sum_{l=1}^{\infty} (l+1)\, 2^l \bar{\varepsilon}(L)^l\, \bar{\varepsilon}_{\mu\mu}(L) + \sum_{l=1}^{\infty} l(l+1)\, 2^l \bar{\varepsilon}(L)^{l-1}\, \bar{\varepsilon}_{\mu\mu}(L)\, 2K_0$$

which also converges to 0 as $L \to \infty$. This shows the lemma.

Now we want to use Lemmas 8 and 9 to show Theorem 2. Using these lemmas, we can choose a sufficiently large $L_1 \geqslant L_0$ such that if $L \geqslant L_1$ then whenever (P_{2K}) holds so does (P_K). Let p_0 and $\tilde{m}_0$ be chosen arbitrarily in $(0, p_c)$ and $(-\infty, m_p)$, respectively. Recall that if (69) fails then (P_K) holds. On the other hand, if (69) holds then the bootstrapping argument "(P_{2K}) implies (P_K)" can be applied. Since the sum over (y, m) of the left hand side of any one of (74)–(78) is finite at $(p_0, \tilde{m}_0)$, there exists a finite set S (depending upon $p_0, \tilde{m}_0$) such that (P_K) holds for every $(y, m) \notin S$. By the dominated convergence theorem, the functions on the left-hand sides of (74)–(78) are continuous with respect to p and $\tilde{m}$ for every $p < p_0$, $\tilde{m} < \tilde{m}_0$; therefore, their maxima over $(y, m) \in S$ are also continuous in this domain. Thus they must be bounded by the gap created by the bootstrapping argument "(P_{2K}) implies (P_K)" for every $p \in (0, p_0]$ and $\tilde{m} \leqslant \tilde{m}_0$. But p_0 and $\tilde{m}_0$ can be chosen arbitrarily in $(0, p_c)$ and $(-\infty, m_p)$, respectively, so (P_K) holds for all $p_0 < p_c$, $\tilde{m}_0 < m_p$. In addition, using the monotone convergence theorem, we can show that these functions are left continuous at $p = p_c$, $\tilde{m} = m_p$; therefore, (P_K) holds at these values as well. Notice that the poof of Lemma 9 is valid even for $p = p_c$, $\tilde{m} = m_p$, so (60), (61) hold. Then (50) and (51) follow from Lemmas 6 and 7. This completes the proof of Theorem 2.

4. PROOF OF LEMMA 2

In this section we prove Lemma 2 using the fractional derivative method developed by Hara and Slade.[13] This method exploits the following integral representation of the ε-fractional derivative for every $\varepsilon \in (0, 1)$: If $f(z) = \sum_{n=0}^{\infty} a_n e^{zn}$ is well defined for every z with $\mathrm{Re}(z) < m_p$ then

$$\delta_z^\varepsilon f(z) = C_\varepsilon \int_0^\infty \partial_z f(z - \lambda^{1/(1-\varepsilon)})\, d\lambda \tag{85}$$

where C_ε is a constant depending only on ε; furthermore, if $a_n \geqslant 0$ then (85) holds for $z = m_p$ as well. We now give a proof for (85) following ref. 13. First we note that

$$n^{\varepsilon - 1} = \frac{1}{(1-\varepsilon)\, \Gamma(1-\varepsilon)} \int_0^\infty \exp(-n\lambda^{1/(1-\varepsilon)})\, d\lambda$$

We then have

$$\sum_{n=0}^{\infty} n^{\varepsilon} a_n \exp(nz) = \sum_{n=0}^{\infty} n^{\varepsilon-1} n a_n \exp(z)$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1-\varepsilon)\,\Gamma(1-\varepsilon)} \int_0^{\infty} \exp(-n\lambda^{1/(1-\varepsilon)})\,d\lambda\, na_n \exp(nz)$$

$$= \frac{1}{(1-\varepsilon)\,\Gamma(1-\varepsilon)} \int_0^{\infty} \sum_{n=0}^{\infty} na_n \exp(nz - n\lambda^{1/(1-\varepsilon)})\,d\lambda$$

$$= \frac{1}{(1-\varepsilon)\,\Gamma(1-\varepsilon)} \int_0^{\infty} \partial_z f(z-\lambda^{1/(1-\varepsilon)})\,d\lambda$$

where in the third line we have used Fubini's theorem to interchange the order of summation and integral. Notice that if $a_n \geqslant 0$ then Fubini's theorem can be applied for $z = m_p$ as well. This proves (85) with $C_\varepsilon = 1/(1-\varepsilon)\,\Gamma(1-\varepsilon)$.

We next apply (85) to prove Lemma 2. To this end, observe that

$$\sum_{(x,n)} n^{\varepsilon} |x|^2 \, |\Psi_c(x,n)| \, e^{nm_p} \leqslant \sum_{(x,n)} n^{\varepsilon} |x|^2 \sum_{l=0}^{\infty} |\Psi_l(x,n)| \, e^{nm_p}$$

$$\leqslant \sum_{(x,n)} n^{\varepsilon} |x|^2 \sum_{l=0}^{\infty} E_l(x,n)\, e^{nm_p}$$

$$= \sum_{\mu=1}^{d} \sum_{l=0}^{\infty} \delta_z^{\varepsilon} \partial_{\mu\mu} \hat{E}_l(0, m_p) \tag{86}$$

By (85) the right-hand side of the above is equal to

$$C_\varepsilon \int_0^{\infty} \sum_{\mu=1}^{d} \sum_{l=0}^{\infty} \partial_{z\mu\mu} \hat{E}_l(0, m_p - \lambda^{1/(1-\varepsilon)})\,d\lambda \tag{87}$$

Similarly, we have

$$\sum_{(x,n)} n^{1+\varepsilon} \, |\Psi_c(x,n)| \, e^{nm_p} \leqslant C_\varepsilon \int_0^{\infty} \sum_{l=0}^{\infty} \partial_{zz} \hat{E}_l(0, m_p - \lambda^{1/(1-\varepsilon)})\,d\lambda \tag{88}$$

However, the integral from 1 to ∞ of the integrand appearing in either (87) or (88) is finite. To see this, we use the following argument in which the function $A(z)=\sum_{n=1}^{\infty} A_n \exp(zn)$ may be replaced by any one of these integrands:

$$\begin{aligned}
&\int_1^{\infty} A(m_p-\lambda^{1/(1-\varepsilon)})\, d\lambda \\
&\quad =\int_1^{\infty} \sum_{n=1}^{\infty} A_n \exp\{(m_p-\lambda^{1/(1-\varepsilon)})\, n\}\, d\lambda \\
&\quad =\int_1^{\infty} \sum_{n=1}^{\infty} A_n \exp\{(m_p-1)n\} \exp\{(1-\lambda^{1/(1-\varepsilon)})\, n\}\, d\lambda \\
&\quad \leqslant\int_1^{\infty} \sum_{n=1}^{\infty} A_n \exp\{(m_p-1)n\} \exp(1-\lambda^{1/1-\varepsilon})\, d\lambda \\
&\quad = A(m_p-1)\int_1^{\infty} \exp(1-\lambda^{1/(1-\varepsilon)})\, d\lambda
\end{aligned}$$

Clearly, $\int_1^{\infty} \exp(1-\lambda^{1/(1-\varepsilon)}\, d\lambda<\infty$. Furthermore, by inserting (35) and (36) in (87) and (88) and then applying (76)–(78), one can see that $A(m_p-1)<\infty$ since the right-hand sides of (76)–(78) can be made as small as possible by choosing sufficiently large L. So $\int_1^{\infty} A(m_p-\lambda^{1/(1-\varepsilon)})\, d\lambda<\infty$.

Thus it suffices to prove (87) and (88) with the upper limit ∞ of these integrals replaced by 1. By inserting (35) and (36) in (87) and (88) and then applying (76)–(78) as before, we see that Lemma 2 will follow from

$$\int_0^1 \partial_{\mu\mu} \hat{T}^{(\alpha)}_{(y,m)}(0, m_p-\lambda^{1/(1-\varepsilon)})\, d\lambda<\infty \tag{89}$$

$$\int_0^1 \partial_z \hat{T}^{(\alpha)}_{(y,m)}(0, m_p-\lambda^{1/(1-\varepsilon)})\, d\lambda<\infty \tag{90}$$

$$\int_0^1 \tilde{Q}^{(\alpha)}_{z\mu\mu}(0, m_p-\lambda^{1/(1-\varepsilon)})\, d\lambda<\infty \tag{91}$$

$$\int_0^1 \tilde{Q}^{(\alpha)}_{zz}(0, m_p-\lambda^{1/(1-\varepsilon)})\, d\lambda<\infty \tag{92}$$

for every $\varepsilon\in(0, 1/2)$, $(y, m)\in Z^d\times Z$, and $\alpha=1, 2, 3$. We only prove this for the case of $\alpha=1$, since proofs for the remaining cases are similar and therefore are omitted. Applying Lemmas 3–5 to the integrands of these integrals, we have

$$\int_0^1 \partial_{\mu\mu} \hat{T}^{(1)}_{(y,m)}(0, m_p - \lambda^{1/(1-\varepsilon)})\, d\lambda$$

$$\leqslant c_7 \int_0^1 \iint |\partial_{\mu\mu} \hat{\Psi}(k, m_p + it - \lambda^{1/(1-\varepsilon)})| \cdot |\hat{\Psi}(k, m_p + it)|^2\, dk\, dt\, d\lambda \qquad (93)$$

$$\int_0^1 \partial_z \hat{T}^{(1)}_{(y,m)}(0, m_p - \lambda^{1/(1-\varepsilon)})\, d\lambda$$

$$\leqslant c_8 \int_0^1 \iint |\partial_z \hat{\Psi}(k, m_p + it - \lambda^{1/(1-\varepsilon)})| \cdot |\hat{\Psi}(k, m_p + it)|^2\, dk\, dt\, d\lambda \qquad (94)$$

$$\int_0^1 \tilde{Q}^{(1)}_{z\mu\mu}(0, m_p - \lambda^{1/(1-\varepsilon)})\, d\lambda$$

$$\leqslant c_9 \int_0^1 \iint |\partial_z \hat{\Psi}(k, m_p + it - \lambda^{1/(1-\varepsilon)})| \cdot |\partial_{\mu\mu} \hat{\Psi}(k, m_p + it)|\, dk\, dt\, d\lambda \qquad (95)$$

$$\int_0^1 \tilde{Q}^{(1)}_{zz}(0, m_p - \lambda^{1/(1-\varepsilon)})\, d\lambda$$

$$\leqslant c_{10} \int_0^1 \iint |\partial_z \hat{\Psi}(k, m_p + it - \lambda^{1/(1-\varepsilon)})| \cdot |\partial_z \hat{\Psi}(k, m_p + it)|\, dk\, dt\, d\lambda \qquad (96)$$

Computing the derivatives of $\hat{\Psi}(k, z)$ from the renewal equation (15) and applying the infrared bound (11), we observe that the right-hand sides of (93) and (95) are bounded above by

$$\text{const} \cdot \int_0^1 \iint |k|^2\, |1 - \exp(-\lambda^{1/(1-\varepsilon)} + it)\, \hat{D}_L(k)|^{-5}\, dk\, dt\, d\lambda$$

and the right-hand sides of (94) and (96) are bounded above by

$$\text{const} \cdot \int_0^1 \iint |1 - \exp(-\lambda^{1/(1-\varepsilon)} + it)\, \hat{D}_L(k)|^{-4}\, dk\, dt\, d\lambda$$

but these integrals are finite if $0 < \varepsilon < 1/2$ and $d > 4$.

5. PROOF OF THEOREM 3

In this section we use the fractional derivative method to show Theorem 3. The fractional derivative method will play a role through a kind of Abelian–Tauberian argument described in the following lemma, which shows that the asymptotic form of the power series is related to the asymptotic behavior of its coefficients.

Lemma 10. Let $\varepsilon \in (0, 1)$, and let $R > 0$ be the radius of convergence of $f(w) \equiv \sum_{n=0}^{\infty} a_n w^n$.

(i) Suppose that $B_1(\varepsilon) \equiv \sum_{n=0}^{\infty} n^{\varepsilon} |a_n| R^n < \infty$. Then for any $|w| \leqslant R$,

$$|f(w) - f(R)| \leqslant 2^{1-\varepsilon} B_1(\varepsilon) R^{-\varepsilon} |R - w|^{\varepsilon} \tag{97}$$

(ii) Suppose that $B_2(\varepsilon) \equiv \sum_{n=0}^{\infty} n^{1+\varepsilon} |a_n| R^n < \infty$. Then for any $|w| \leqslant R$,

$$|f(w) - f(R) - f'(R)(w - R)| \leqslant \frac{2^{1-\varepsilon}}{1+\varepsilon} R^{-1-\varepsilon} B_2(\varepsilon) |R - w|^{1+\varepsilon} \tag{98}$$

(iii) Suppose that for any $|w| < R$, $|f(w)| \leqslant \text{const} \cdot |R - w|^{\varepsilon - 2}$. Then $|a_n| \leqslant O(R^{-n} n^{1-\alpha})$ for any $\alpha < \varepsilon$.

(iv) Suppose that for any $|w| < R$, $|f'(w)| \leqslant \text{const} \cdot |R - w|^{\varepsilon - 2}$. Then $|a_n| \leqslant O(R^{-n} n^{-\alpha})$ for any $\alpha < \varepsilon$.

The proof of this lemma can be found in the proofs of Lemmas 3.2 and 3.3 of ref. 13. We remark that the proof of lemma 3.3 of ref. 13 should also work for $|w| < R$ instead of $|w| \leqslant R$.

In addition we need the following, which will be useful in simplifying calculations involving the inverse of $[1 + \hat{\Psi}_c(k, z)]$.

Lemma 11. Let ε, R, f, and $B_2(\varepsilon)$ be defined as in Lemma 10. Assume that in addition to $B_2(\varepsilon) < \infty$ there exist constants $0 < c_1 < c_2$ such that $c_1 \leqslant |f(w)| \leqslant c_2$ for every $|w| \leqslant R$. Then

$$\frac{1}{f(w)} = \frac{1}{f(R)} + \frac{f'(R)}{[f(R)]^2} (R - w) + O(|R - w|^{1+\varepsilon}) \tag{99}$$

$$\partial_w \frac{1}{f(w)} = -\frac{f'(R)}{[f(R)]^2} + O(|R - w|^{\varepsilon}) \tag{100}$$

Proof. From part (ii) of Lemma 10 we have for any w with $|w| < R$

$$f(w) = f(R) - f'(R)(R - w) + O(|R - w|^{1+\varepsilon}) \tag{101}$$

Observe also from our assumptions that

$$\frac{1}{f(w)} = \frac{1}{f(R)} + O(|R - w|)$$

Therefore,

$$\begin{aligned}\frac{1}{f(w)}-\frac{1}{f(R)}&=\frac{f(R)-f(w)}{f(w)\,f(R)}\\&=\frac{f'(R)(R-w)}{f(w)\,f(R)}+\frac{O(|R-w|^{1+\varepsilon})}{f(w)\,f(R)}\\&=\frac{f'(R)(R-w)}{f(R)}\left[\frac{1}{f(R)}+O(|R-w|)\right]+O(|R-w|^{1+\varepsilon})\\&=\frac{f'(R)(R-w)}{[f(R)]^2}+O(|R-w|^{1+\varepsilon})\end{aligned}\tag{102}$$

where in the last two lines we have used the assumption $c_1 \leqslant f(w) \leqslant c_2$. This shows (99). We next apply part (i) of Lemma 10 on $f'(w)$ to show (100) as follows:

$$\begin{aligned}\partial_w \frac{1}{f(w)}&=\frac{-f'(w)}{[f(w)]^2}\\&=\{-f'(R)+O(|R-w|^{\varepsilon})\}\left\{\frac{1}{f(R)}+O(|R-w|)\right\}^2\\&=-\frac{f'(R)}{[f(R)]^2}+O(|R-w|^{\varepsilon})\end{aligned}\tag{103}$$

To apply the above lemmas it would be more convenient to work with $w=e^z$ instead of z; therefore, throughout this section we will use respectively $\tilde{\varphi}(w)$, $\tilde{\Psi}_c(w)$ to denote $\hat{\varphi}(z)$, $\hat{\Psi}_c(z)$, where w and z are related by $w=e^z$. We point out that $[1+\tilde{\Psi}_c(w)]$ satisfies the conditions of the above lemma for every $p\in(0, p_c]$ and $\varepsilon\in(0, 1/2)$ by Lemma 2 and (50). Due to this, we fix $\varepsilon\in(0, 1/2)$ for the rest of this section.

We now use Lemmas 10 and 11 to prove Theorem 3. Our approach to Theorem 3 is to compare $\tilde{\varphi}(k, w)$ with

$$H(k, w)\equiv\frac{1}{A_0(w_p-w)+A_1\,|k|^2}\tag{104}$$

where $w_p=e^{m_p}$ and A_0, A_1 are two positive numbers that will be chosen later depending upon $p\in(0, p_c]$. Let $H_n(k)$ be the nth coefficient of the MacLaurin series of $H(k, w)$. Write

$$H(k, w)=\left[A_0 w_p\left(1+\frac{A_1}{A_0 w_p}\,|k|^2\right)\right]^{-1}\left[1-\frac{w}{w_p(1+A_1/A_0 w_p\,|k|^2)}\right]^{-1}$$

and use geometric series expansion to obtain

$$H_n(k) = \frac{1}{A_0 w_p} w_p^{-n} \left(1 + \frac{A_1}{A_0 w_p} |k|^2\right)^{-n-1} \tag{105}$$

Then simple calculations show that (a)–(c) of Theorem 3 hold for $H_n(k)$ instead of $Z_n(k)$ as follows:

(a′) $H_n(0) = \dfrac{1}{A_0 w_p} w_p^{-n}$

(b′) $[H_n(0)]^{-1} \nabla_k^2 H_n(0) = -(n-1) \dfrac{2dA_1}{A_0 w_p}$

(c′) $[H_n(0)]^{-1} H_n\left(\dfrac{k}{\sqrt{n}}\right) \to \exp\left(-\dfrac{A_1}{A_0 w_p} |k|^2\right)$

Thus by (iii) and (iv) of Lemma 10, parts (a) and (b) of Theorem 3 will follow if we can choose A_0 and A_1 such that

$$|\partial_w[\tilde{\varphi}(0, w) - H(0, w)]| \leqslant \text{const} \cdot |w_p - w|^{\varepsilon-2} \tag{106}$$

$$|\nabla_k^2[\tilde{\varphi}(0, w) - H(0, w)]| \leqslant \text{const} \cdot |w_p - w|^{\varepsilon-2} \tag{107}$$

for every w with $|w| < w_p$.

We then use (106) to choose A_0 as follows. We define $A_2(k, w)$ for every $k \in [-\pi, \pi]^d$ and w with $|w| < w_p$ by

$$A_2(k, w) = \frac{1}{1 + \hat{\Psi}_c(k, w)} - pp_L^{-1} w \hat{D}_L(k) - A_0(w_p - w) - A_1 |k|^2 \tag{108}$$

Then

$$\tilde{\varphi}(k, w) = \frac{1}{A_0(w_p - w) + A_1 |k|^2 + A_2(k, w)} \tag{109}$$

$$\tilde{\varphi}(k, w) - H(k, w) = -A_2(k, w)\, \tilde{\varphi}(k, w)\, H(k, w) \tag{110}$$

and

$$\begin{aligned} \partial_w[\tilde{\varphi}(k, w) - H(k, w)] = &- \partial_w A_2(k, w)\, \tilde{\varphi}(k, w)\, H(k, w) \\ &- A_2(k, w)[\partial_w \tilde{\varphi}(k, w)]\, H(k, w) \\ &- A_2(k, w)\, \tilde{\varphi}(k, w)\, \partial_w H(k, w) \end{aligned} \tag{111}$$

With straightforward calculations we can see from the infrared bound (11) that

$$|\tilde{\varphi}(k, w)| = O(|w_p - w|^{-1}) \tag{112}$$

$$|\partial_w \tilde{\varphi}(k, w)| = O(|w_p - w|^{-2}) \tag{113}$$

Thus (106) will follow from (111) if we can choose A_0 and A_1 such that

$$|A_2(0, w)| = O(|w_p - w|^{1+\varepsilon}) \tag{114}$$

$$|\partial_w A_2(0, w)| = O(|w_p - w|^{\varepsilon}) \tag{115}$$

for every $|w| < w_p$ since clearly $H(0, w) = O(|w_p - w|^{-1})$ and $\partial_w H(0, w) = O(|w_p - w|^{-2})$. As we mentioned earlier, $[1 + \tilde{\Psi}_c(k, w)]$ satisfies the conditions of Lemma 11, so

$$\frac{1}{1 + \tilde{\Psi}_c(0, w)} = \frac{1}{1 + \tilde{\Psi}_c(0, w_p)} + \frac{\partial_w \tilde{\Psi}_c(0, w_p)(w_p - w)}{[1 + \tilde{\Psi}_c(0, w_p)]^2} + O(|w_p - w|^{1+\varepsilon}) \tag{116}$$

Replace $k = 0$ in (108), add and subtract the same term $pp_L^{-1} w_p$, and then apply the above identity to obtain

$$\begin{aligned} A_2(0, w) &= \frac{1}{1 + \hat{\Psi}_c(0, w)} - pp_L^{-1} w - A_0(w_p - w) \\ &= \frac{1}{1 + \tilde{\Psi}_c(0, w_p)} + \frac{\partial_w \tilde{\Psi}_c(0, w_p)}{[1 + \tilde{\Psi}_c(0, w_p)]^2}(w_p - w) + O(|w_p - w|^{1+\varepsilon}) \\ &\quad - pp_L^{-1} w_p + pp_L^{-1}(w_p - w) - A_0(w_p - w) \end{aligned}$$

By (57), $[1 + \tilde{\Psi}_c(0, w_p)]^{-1} - pp_L^{-1} w_p = 0$, so we can choose

$$A_0 = pp_L^{-1} + \frac{\partial_w \tilde{\Psi}_c(0, w_p)}{[1 + \tilde{\Psi}_c(0, w_p)]^2} \tag{117}$$

to obtain the desired estimate (114). Since $\partial_w \tilde{\Psi}_c(0, w_p)/[1 + \tilde{\Psi}_c(0, w_p)]^2$ can be as small as possible by taking sufficiently large L, we have $A_0 > 0$ if $pp_L^{-1} > 1$; otherwise, we can follow the same argument discussed in ref. 21 to show $A_0 > 0$ if $pp_L^{-1} \leqslant 1$. With such a choice of A_0 we can use (100) to obtain (115).

The choice of A_1 can be selected from (107) similarly as follows. From (109) we have

$$\begin{aligned} &|\nabla_k^2 \tilde{\varphi}(0, w) - \nabla_k^2 H(0, w)| \\ &= \left| \frac{-2dA_1 - \nabla_k^2 A_2(0, w)}{[A_0(w_p - w) + A_2(0, w)]^2} - \frac{-2dA_1}{[A_0(w_p - w)]^2} \right| \\ &= \left| \frac{-\nabla_k^2 A_2(0, w)[A_0(w_p - w)]^2 + 2dA_1[A_2(0, w)]^2 + 2dA_1 2A_0(w_p - w) A_2(0, w)}{[A_0(w_p - w) + A_2(0, w)]^2 [A_0(w_p - w)]^2} \right| \end{aligned} \tag{118}$$

for every w with $|w| < w_p$. By (114) the sum of the last two terms in the numerator of the above equation is $O(|w_p - w|^{2+\varepsilon})$. Furthermore, since $[A_0(w_p - w) + A_2(0, w)]^{-1}$ is $O(|w_p - w|^{-1})$, we need to choose A_1 in such a way that $\nabla_k^2 A_2(0, w)$ is $O(|w_p - w|^{\varepsilon})$; if this is so, then (107) follows from (118). But from Lemmas 2 and 10 we have

$$\nabla_k^2 \tilde{\Psi}_c(0, w) = \nabla_k^2 \tilde{\Psi}_c(0, w_p) + O(|w_p - w|^{\varepsilon}) \tag{119}$$

Hence,

$$\begin{aligned} \nabla_k^2 A_2(0, w) &= \frac{-\nabla_k^2 \tilde{\Psi}_c(0, w)}{[1 + \tilde{\Psi}_c(0, w)]^2} - pp_L^{-1} w \nabla_k^2 \hat{D}_L(0) - 2dA_1 \\ &= \frac{-\nabla_k^2 \tilde{\Psi}_c(0, w_p) + O(|w_p - w|^{\varepsilon})}{[1 + \tilde{\Psi}_c(0, w)]^2} - pp_L^{-1} w \nabla_k^2 \hat{D}_L(0) - 2dA_1 \end{aligned} \tag{120}$$

After replacing $[1 + \tilde{\Psi}_c(0, w)]^{-1}$ by $[1 + \tilde{\Psi}_c(0, w_p)]^{-1}$ in the right-hand side of the above identity and compensating this replacement by a term of $O(|w_p - w|)$, we then choose

$$A_1 = -\frac{1}{2d} \left[\frac{\nabla_k^2 \tilde{\Psi}_c(0, w_p)}{[1 + \tilde{\Psi}_c(0, w_p)]^2} + pp_L^{-1} w_p \nabla_k^2 \hat{D}_L(0) \right] \tag{121}$$

to obtain from (120) the desired estimate $O(|w_p - w|^{\varepsilon})$ for $\nabla_k^2 A_2(0, w)$. Since $\nabla_k^2 \tilde{\Psi}_c(0, w_p)/[1 + \tilde{\Psi}_c(0, w_p)]^2$ can be as small as possible by choosing large enough L and since $-\nabla_k^2 \hat{D}_L(0) \geqslant \text{const} \cdot L^2$, by (49), we have $A_1 > 0$ if $pp_L^{-1} w_p > 1$. On the other hand, if $pp_L^{-1} w_p \leqslant 1$, then we can use the same argument discussed in ref. 21 to show $A_1 > 0$. Thus with A_0 as in (117) and

A_1 as in (121) we only need to show part (c) of Theorem 3. We claim that for $\alpha < \varepsilon/2$,

$$|A_2(k, w_p)| = O(|k|^{2+2\alpha}) \tag{122}$$

$$\left|A_2\left(\frac{k}{\sqrt{n}}, w\right) - A_2\left(\frac{k}{\sqrt{n}}, w_p\right) - A_2(0, w)\right| = O(n^{-\alpha})\, O(|w_p - w|) \tag{123}$$

$$\left|\partial_w A_2\left(\frac{k}{\sqrt{n}}, w\right) - \partial_w A_2(0, w)\right| = O(n^{-\alpha}) \tag{124}$$

for $\alpha < \varepsilon/2$ and $|w| < w_p$. Assuming the above estimates, we now prove Theorem 3(c). We define

$$H^0(k, w) \equiv \frac{1}{A_0(w_p - w) + A_1\,|k|^2 + A_2(k, w_p)} \tag{125}$$

Notice that $H^0(k, w)$ looks similar to $\tilde{\varphi}(k, w)$ except that $A_2(k, w_p)$ [instead of $A_2(k, w)$] appears in the denominator of $H^0(k, w)$. Then

$$\begin{aligned}
&\partial_w[\tilde{\varphi}(k, w) - H^0(k, w)] \\
&\quad = -\partial_w A_2(k, w)\, \tilde{\varphi}(k, w)\, H^0(k, w) \\
&\qquad - [A_2(k, w) - A_2(k, w_p)][\partial_w \tilde{\varphi}(k, w)]\, H^0(k, w) \\
&\qquad - [A_2(k, w) - A_2(k, w_p)]\, \tilde{\varphi}(k, w)\, \partial_w H^0(k, w)
\end{aligned} \tag{126}$$

It is not hard to show that for sufficiently large n,

$$\left|H^0\left(\frac{k}{\sqrt{n}}, w\right)\right| \leqslant O(|w_p - w|^{-1}) \tag{127}$$

$$\left|\partial_w H^0\left(\frac{k}{\sqrt{n}}, w\right)\right| \leqslant O(|w_p - w|^{-2}) \tag{128}$$

Applying the infrared bounds (112) and (113) and the bounds (127) and (128), we can see from (126) that

$$\begin{aligned}
&\left|\partial_w \left\{\tilde{\varphi}\left(\frac{k}{\sqrt{n}}, w\right) - H^0\left(\frac{k}{\sqrt{n}}, w\right)\right\}\right| \\
&\quad = \left|\partial_w A_2\left(\frac{k}{\sqrt{n}}, w\right)\right| O(|w_p - w|^{-2}) + \left|A_2\left(\frac{k}{\sqrt{n}}, w\right)\right. \\
&\qquad \left. - A_2\left(\frac{k}{\sqrt{n}}, w\right)\right| O(|w_p - w|^{-3})
\end{aligned}$$

$$\leqslant \left| \partial_w A_2\left(\frac{k}{\sqrt{n}}, w\right) - \partial_w A_2(0, w) \right| O(|w_p - w|^{-2})$$

$$+ |\partial_w A_2(0, w)| \, O(|w_p - w|^{-2})$$

$$+ \left| A_2\left(\frac{k}{\sqrt{n}}, w\right) - A_2\left(\frac{k}{\sqrt{n}}, w\right) - A_2(0, w) \right| O(|w_p - w|^{-3})$$

$$+ |A_2(0, w)| \, O(|w_p - w|^{-3})$$

for every $|w| < w_p$, $k \in [-\pi, \pi]^d$, and sufficiently large n. By (123) and (124) the first and third terms are $O(n^{-\alpha}) \, O(|w_p - w|^{-2})$, and by (114) and (115) the second and fourth terms are $O(|w_p - w|^{\varepsilon - 2})$.

It then follows from Lemma 10(v) that

$$|Z_n(k/\sqrt{n}) - H_n^0(k/\sqrt{n})| = O(n^{-\alpha}) \, w_p^{-n-1}$$

Thus part (c) of Theorem 3 follows by applying a simple central limit theorem type of proof on $H_n^0(k/\sqrt{n})$.

We now return to our claim of (122)–(124). From the definition of $A_2(k, w)$ and our choice of A_1 we have

$$A_2(k, w_p) = \left\{ \frac{1}{1 + \tilde{\Psi}_c(k, w_p)} - \frac{1}{1 + \tilde{\Psi}_c(0, w_p)} - \frac{1}{2d} \frac{\nabla_k^2 \tilde{\Psi}_c(0, w_p)}{[1 + \tilde{\Psi}_c(0, w_p)]^2} |k|^2 \right\}$$
$$+ \left\{ p p_L^{-1} w_p \left[1 - \hat{D}_L(k) - \frac{1}{2d} \nabla_k^2 \hat{D}_L(0) \, |k|^2 \right] \right\} \qquad (129)$$

Using symmetry, we have

$$\left| \tilde{\Psi}_c(0, w_p) - \tilde{\Psi}_c(k, w_p) - \frac{1}{2d} \nabla_k^2 \tilde{\Psi}_c(0, w_p) \, |k|^2 \right|$$

$$= \left| \sum_{(x, m)} e^{m z} \Psi_c(x, m) \left[1 - \cos(k \cdot x) - \frac{1}{2} (k \cdot x)^2 \right] \right|$$

$$\leqslant \text{const} \cdot \sum_{(x, m)} e^{m m_p} \, |\Psi_c(x, m)| \cdot |k \cdot x|^{2+\varepsilon}$$

$$\leqslant \text{const} \cdot \sum_{(x, m)} e^{m m_p} \, |\Psi_c(x, m)| \cdot |k|^{2+\varepsilon} m^{2+\varepsilon}$$

$$\leqslant \text{const} \cdot |k|^{2+\varepsilon} \qquad (130)$$

where in the last two inequalities we used

$$|k \cdot x|^{2+\varepsilon} |\Psi_c(x, m)| \leqslant \text{const} \cdot |k|^{2+\varepsilon} m^{2+\varepsilon} |\Psi_c(x, m)| \tag{131}$$

(since the bond density is of finite range) and Lemma 2, respectively. Then applying the identity

$$\frac{1}{a} - \frac{1}{b} = \frac{b-a}{b^2} + \frac{[b-a]^2}{ab^2}$$

with $a = 1 + \tilde{\Psi}_c(k, w_p)$ and $b = 1 + \tilde{\Psi}_c(0, w_p)$, we have

$$\frac{1}{1 + \tilde{\Psi}_c(k, w_p)} - \frac{1}{1 + \tilde{\Psi}_c(0, w_p)} = \frac{\tilde{\Psi}_c(0, w_p) - \tilde{\Psi}_c(k, w_p)}{[1 + \tilde{\Psi}_c(0, w_p)]^2} + O(|k|^4) \tag{132}$$

since $a, b \geqslant 1/2$ and $b - a = O(|k|^2)$. It follows from (130) and (132) that the first term of (129) is $O(|k|^{2+\varepsilon})$. The proof for the $O(|k|^{2+\varepsilon})$ behavior of the second term of (129) can be shown in the same way as in the proof of (130). This shows (122).

It remains to show (123, 124). We have

$$\begin{aligned}
&\left| \partial_w \tilde{\Psi}_c(0, w) - \partial_w \tilde{\Psi}_c\left(\frac{k}{\sqrt{n}}, w\right) \right| \\
&\quad = \left| \sum_{(x,m)} m e^{mz} \Psi_c(x, m) \left[1 - \cos\left(\frac{k \cdot x}{\sqrt{n}}\right) \right] \right| \\
&\quad \leqslant \text{const} \cdot \sum_{(x,m)} m e^{mm_p} |\Psi_c(x, m)| \left| \frac{k \cdot x}{\sqrt{n}} \right|^{\varepsilon}
\end{aligned} \tag{133}$$

since $|1 - \cos t| = O(|t|^{\varepsilon})$. Furthermore, since the bond density is of finite range, we have

$$|k \cdot x/\sqrt{n}|^{\varepsilon} |\Psi_c(x, m)| \leqslant \text{const} \cdot n^{-\varepsilon/2} m^{\varepsilon} |\Psi_c(x, m)| \tag{134}$$

By Lemma 2, the right-hand side of (133) is $O(n^{-\varepsilon/2})$. This shows (124). We then integrate (124) to get (123).

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